Conservation of wave action and radiative energy transfer in space- and time-varying media

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The conservation of the (standard) wave-action density inherent in wave propagation in a space- and time-varying medium is shown to require a continuity-type equation involving the time derivative of the wave frequency and the divergence [in ($\mathbf{r}$,$\mathbf{k}$) phase space] of the group velocity. Such a continuity-type equation embodies the condition for the validity of the equation of radiative transfer. A comparative analysis of three different definitions of wave-action density is also made. [S1063-651X(98)06605-7]

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I. INTRODUCTION

In the analysis of wave propagation in space- and time-varying media, for which the geometrical optics approximation is assumed to be valid, one encounters two types of wave kinetic equations, namely, the kinetic equation for the transport of the spectral energy density $W(\mathbf{k},\mathbf{r},t)$ [1,2] and the kinetic equation for the transport of the wave-action density $J(\mathbf{k},\mathbf{r},t)$ [3,4]. On viewing the waves from a quantum viewpoint, the latter equation is the same as the one governing the occupation number of a mode [5–7]. Here $\mathbf{r}$ represents the point of maximum constructive interference at time t for a wave packet centered on the local wave vector $\mathbf{k}(\mathbf{r},t)$.

The quite different approaches adopted in the derivation of the two wave kinetic equations make it somewhat arduous to assess the relative physical contents of the two. In particular, whereas the kinetic equation for the wave-energy density accounts for focusing effects [8], such as caustics and focal points, the kinetic equation for the wave-action density is free from the effects of ray focusing and is conveniently used to derive the equation of transfer for the specific intensity of the radiation [2,7].

The definition of wave action for a space- and time-varying medium needs to be examined. Whereas in Ref. [4] a definition is adopted that generalizes the standard definition of wave action in a uniform (i.e., homogeneous and stationary) medium, namely, the wave action is the spectral energy density divided by the frequency, the latter being the solution of the dispersion equation, in Ref. [9] an exponential-type definition for the wave action is proposed that differs, for the case of a nonuniform medium, from that given in Ref. [4]. A third definition appears in Refs. [10, 11], where the wave action is defined in terms of the derivative of the Lagrangian density with respect to frequency. With regard to these different definitions, one should note that the physical interpretation of the equation of radiative transfer as describing the ray evolution of the specific intensity of radiation requires the standard definition of action to be adopted even in the case of a space- and time-varying medium [2,7].

In this paper, in Sec. II, the conservation of the (standard) wave-action density is examined and it is shown that a continuity-type equation involving the time derivative of the wave frequency and the phase-space divergence of the group velocity must be satisfied. The relevance of such a continuity-type equation is assessed in the light of the equation of radiative transfer. In Sec. III a comparative analysis of three different definitions of wave action is made. A few conclusions are finally given in Sec. IV.

II. CONSERVATION OF THE WAVE ACTION, ADIABATICITY CONDITION, AND EQUATION OF RADIATIVE TRANSFER

The transport equation for the wave-energy density $W(\mathbf{k},\mathbf{r},t)$ in a space- and time-varying medium, which follows directly from the geometrical optics equation for the wave amplitude, can be expressed in the form (for simplicity, let us disregard any process of emission and absorption of radiation) [1,9,2]

$$\frac{dW(\mathbf{k},\mathbf{r},t)}{dt}=-(\boldsymbol{\nabla}\cdot\mathbf{v}_g)W(\mathbf{k},\mathbf{r},t). \tag{1}$$

Equation (1), which refers to a single geometrical optics mode, contains the total time derivative along the ray trajectories in ($\mathbf{r}$,$\mathbf{k}$) phase space, on its left-hand side, and the total divergence operator, on its right-hand side, namely,

$$\frac{d}{dt}\equiv\frac{\partial}{\partial t}+\dot{\mathbf{k}}\cdot\frac{\partial}{\partial\mathbf{k}}+\dot{\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{r}}, \tag{2a}$$

where $\dot{\mathbf{k}}\equiv d\mathbf{k}/dt$ and $\dot{\mathbf{r}}\equiv d\mathbf{r}/dt$ $(=\mathbf{v}_g$, the group velocity), and

$$\boldsymbol{\nabla}\cdot\equiv\frac{\partial}{\partial\mathbf{r}}\cdot+\mathrm{Tr}\left\{\frac{\partial\mathbf{k}}{\partial\mathbf{r}}\cdot\frac{\partial}{\partial\mathbf{k}}\right\}, \tag{2b}$$

Tr denoting the trace. The right-hand side of Eq. (1) being nonzero, with $\boldsymbol{\nabla}\cdot\mathbf{v}_g$ describing the effects of ray focusing, prevents the wave-energy density from being a constant along a geometrical optics ray and makes Eq. (1) "actually of dubious utility"; cf. Ref. [9], p. 122.

Let us now express the wave-energy density in terms of the standard (Hamiltonian) wave-action density

$$W(\mathbf{k},\mathbf{r},t)=J(\mathbf{k},\mathbf{r},t)\Omega(\mathbf{k},\mathbf{r},t), \tag{3}$$

which is the same relation valid for the harmonic oscillator, the frequency ω of which satisfies the local dispersion relation $\omega=\Omega(\mathbf{k},\mathbf{r},t)$, with $\dot{\mathbf{r}}=\partial\Omega/\partial\mathbf{k}$, $\dot{\mathbf{k}}=-\partial\Omega/\partial\mathbf{r}$, and $\dot{\omega}$

($\equiv d\omega/dt$)$=\partial\Omega/\partial t$ the ray equations. On using Eq. (3) in Eq. (1), one obtains the kinetic equation for the wave-action density

$$\frac{dJ(\mathbf{k},\mathbf{r},t)}{dt}=-\left(\frac{1}{\Omega(\mathbf{k},\mathbf{r},t)}\frac{\partial\Omega(\mathbf{k},\mathbf{r},t)}{\partial t}+\boldsymbol{\nabla}\cdot\mathbf{v}_g\right)J(\mathbf{k},\mathbf{r},t). \tag{4}$$

The transport equation (4) shows that, in general, the classical wave-action density in a generic inhomogeneous and time-varying medium is *not* conserved along a ray in phase space, even in the absence of wave emission and dissipation. Conservation of wave-action density demands that

$$\frac{\partial\Omega(\mathbf{k},\mathbf{r},t)}{\partial t}+\Omega(\mathbf{k},\mathbf{r},t)(\boldsymbol{\nabla}\cdot\mathbf{v}_g)=0, \tag{5}$$

to be referred to as the *adiabaticity condition*. Equation (5) constitutes the main result of this paper. Equation (5), which is trivially satisfied for a uniform (homogeneous and stationary) medium, has the form of a continuity equation, where the ($\mathbf{r}$,$\mathbf{k}$) variations of the ray (vector) field $\mathbf{v}_g$ are connected with the time variations of the frequency (scalar) field Ω. From a geometrical viewpoint Eq. (5) states that no new ray can form at a given point of the configuration space unless the frequency at that point depends on time.

From a quantum perspective, for which the frequency ω represents the energy of the wave quantum, the action density, according to relation (3), amounts to the density (in phase space) of the number of wave quanta and Eq. (4) represents the transport equation for this quantity. One thus expects that, in the absence of emission and absorption processes, the number of wave quanta per unit volume of phase space is constant along the ray: That this is actually the case requires the adiabaticity condition (5) to be satisfied. As a consequence, *the radiation propagating in a nonhomogeneous and time-varying medium can be dealt with in terms of wave quanta only if the medium is such that the adiabaticity condition (5) is satisfied*.

The adiabaticity condition (5) is particularly relevant in the context of the equation of radiative energy transfer. Such an equation is derived from the *equation of conservation* for the wave-action density along a geometrical optics ray on changing the independent variables from $(\mathbf{k},\mathbf{r},t)$ to $(\omega,\hat{\mathbf{s}},s,t)$, with $\hat{\mathbf{s}}\equiv\mathbf{v}_g/|\mathbf{v}_g|$ and s measuring the distance along the ray trajectory in $\mathbf{r}$ space (it suffices that the dependence on position in the new variables only enters via the dependence on s). In the absence of radiation sources and absorption, the equation of radiative transfer reads [2,7]

$$\frac{\partial}{\partial t}\left(\frac{I(\omega,\hat{\mathbf{s}})}{v_g\omega}\right)+\omega^2 n_r^2\frac{\partial}{\partial s}\left(\frac{I(\omega,\hat{\mathbf{s}})}{\omega^3 n_r^2}\right)=0, \tag{6}$$

where n_r is the ray refractive index [7,12] and the specific intensity of the radiation I is defined in terms of the wave-action density J,

$$I(\omega,\hat{\mathbf{s}})\equiv n_r^2\frac{\omega^2}{(2\pi)^3c^2}[\omega J(\mathbf{k}(\omega,\hat{\mathbf{s}}))]. \tag{7a}$$

[$\mathbf{k}(\omega,\hat{\mathbf{s}})$ represents the wave vector corresponding to the frequency ω and to the ray direction $\hat{\mathbf{s}}$.] For the case for which the wave-action density is given by the standard relation (3), the specific intensity of radiation (7a) reduces to its standard definition in terms of the wave-energy density W,

$$I(\omega,\hat{\mathbf{s}})\equiv n_r^2\frac{\omega^2}{(2\pi)^3c^2}W(\mathbf{k}(\omega,\hat{\mathbf{s}})). \tag{7b}$$

In this respect one should note that the equation of radiative transfer is usually obtained, with both wave emission and absorption accounted for, on balancing the rate of change of the number of wave quanta along a geometrical optics ray, as given by the left-hand side of Eq. (4), with the rate of emission and absorption [7,12]. Thus, though implicitly, such a derivation rests on the assumption that the adiabaticity condition (5) is satisfied. The analysis carried out here just points out that the adiabaticity condition (5) is inherent in the equation of radiative transfer.

III. GENERALIZED EXPRESSIONS FOR THE WAVE ACTION: A COMPARATIVE ANALYSIS

As shown in Sec. II, for a generic nonuniform medium the standard wave action [cf. relation (3)] is a conserved quantity along a ray in phase space only provided that the adiabaticity condition (5) is satisfied [cf. Eq. (4)]. One might wonder whether there exist quantities, to be referred to as generalized wave-action densities, that are both conserved along a ray, for a (slowly) space- and time-varying medium, and reduce to the standard wave action (3) in the limit of a uniform medium.

In Ref. [4] a generalized wave-action density is obtained within a description of wave propagation based on the Weyl representation of both the integro-differential tensorial operator relevant to the wave equation, i.e., the dispersion tensor, and the bilinear quantities connected with the autocorrelation of both the electric field and the current source field. In such an approach the wave equation is treated from the outset in the phase space, with the result that the correlation among the variables $(\mathbf{r},t)$ and $(\mathbf{k},\omega)$, that is, the local dispersion relation, as well as the transport equation for the action density along with its definition emerges from solving the exact equation for the phase-space evolution of the electric-field spectral density by means of a perturbative method consistent with the assumptions of geometrical optics. This procedure permits, in particular, one to avoid the ''quasi-plane-wave'' ansatz inherent in the eikonal treatment [1,9,2]. A different quantity is thus obtained, which satisfies a ray-conservation equation even for media not subjected to the adiabaticity condition (5), and is such that it reduces to the standard wave action (3) in the limit of a uniform medium. Such a quantity, taken as a generalized wave-action density, is [4]

$$J^{(MD)}(\mathbf{k},\mathbf{r},t)\equiv\left[\frac{\partial D(\mathbf{k},\omega,\mathbf{r},t)}{\partial\omega}\right]_{\omega=\Omega(\mathbf{k},\mathbf{r},t)}\int\frac{d\omega}{2\pi}\mathcal{W}_E(\mathbf{k},\omega,\mathbf{r},t), \tag{8}$$

where D is the eigenvalue, relevant to the considered mode, of the (Hermitian part of) the dispersion tensor and $4\pi\mathcal{W}_E$ $(\mathbf{k},\omega,\mathbf{r},t)$ is the corresponding eigenvalue of the spectral density of the electric field

$$\langle \mathbf{E}\mathbf{E}^*\rangle(\mathbf{k},\omega;\mathbf{r},t) = \int d^3s \int d\tau \langle \mathbf{E}(\mathbf{r}+\tfrac{1}{2}\mathbf{s},t+\tfrac{1}{2}\tau)\mathbf{E}(\mathbf{r}-\tfrac{1}{2}\mathbf{s},t-\tfrac{1}{2}\tau)\rangle e^{-i(\mathbf{k}\cdot\mathbf{s}-\omega\tau)}, \tag{9a}$$

the angular brackets and the asterisk denoting an ensemble average and the complex conjugate, respectively. The tensor (9a) is diagonal in the same basis as the dispersion tensor [4,2].

For the specific case of a uniform medium,

$$\int \frac{d\omega}{2\pi}\,\mathcal{W}_E(\mathbf{k},\omega;\mathbf{r},t) = \frac{|\mathcal{E}(\mathbf{k})|^2}{4\pi V}, \tag{9b}$$

which is just the spectral electric energy density of the considered mode, with $\mathcal{E}(\mathbf{k})$ the electric field amplitude and V a volume large compared to the cube of the wavelength. The derivation of the result (9b) is outlined in the Appendix. On noting that $\partial D/\partial\omega = 1/\omega R_E$, R_E being the ratio of the electric to the total wave-energy density, the wave action (8) just reduces to $J(\mathbf{k},\mathbf{r},t) = W(\mathbf{k})/\Omega(\mathbf{k})$, as expected in a uniform medium. It is to be noted that the conservation of the wave-action density (8) in a nonuniform medium occurs, in general, independently of the equality (9b).

Within the scheme of the standard geometrical optics [9], an alternative quantity that is constant along the ray and such that it reduces to the standard expression for the wave action in the uniform limit can be obtained on the basis of the wave-energy density [Eq. (1)], namely [9,2],

$$J^{(BB)}(\mathbf{k},\mathbf{r},t) \equiv \frac{W(\mathbf{k},\mathbf{r},t)}{\omega_\infty}\exp\left(\int_{-\infty}^{t} dt'(\nabla\cdot\mathbf{v}_g)\right), \tag{10}$$

where the integration along the ray occurring in the exponential factor is extended to the entire path precedent the point at time t. It is assumed that the medium is such that the adiabaticity condition (5) holds at its boundaries, where $\omega = \omega_\infty$. A feature of definition (10), common also to the standard relation (3), is the occurrence of the wave-energy density $W(\mathbf{k},\mathbf{r},t)$ as obtained from the electric-field amplitude of geometrical optics; the exponential factor $\exp[\int_{-\infty}^{t} dt'(\nabla\cdot\mathbf{v}_g)]$, on the other hand, is the ingredient that guarantees the conservation of the wave action (10) in a space- and time-varying medium. Under the adiabaticity condition (5), it is straightforward to see that expression (10) reduces to the standard relation (3) for the wave action, with the consequence that the corresponding specific intensity of radiation is the standard expression (7b).

Definitions (8) and (10) for the wave action share the feature of being *nonlocal* quantities: Whereas expression (8) requires the evaluation of the autocorrelation of the electric field [cf. Eq. (9a)], which is an integral over space and time of the bilinear product of fields evaluated at different positions and times, expression (10) contains an integral along the ray of the geometrical optics quantity $\nabla\cdot\mathbf{v}_g$. With respect to Eq. (8), the definition (10) has the advantage that it yields the standard wave action (3), as well as the standard specific intensity of radiation (7b), not only in the uniform limit, the same occurring for expression (8), but also in an adiabatic medium for which condition (5) is satisfied. In passing it should be noted that, in this context, the uniform limit is a somewhat trivial limit since in this case the energy density and the action density are proportional to each other and thus are both conserved along the ray [cf. Eqs. (1) and (4)], so that defining a quantity such as the wave action is redundant.

There exists a third definition of wave action, based on an *average* variational treatment of the wave propagation in a medium [11]. As is well known, by suitably writing the Lagrangian density for the electromagnetic field, one can obtain Maxwell's equations as Euler's equations from Hamilton's variational principle. The Lagrangian density for the radiation field is [13]

$$L\left[\mathbf{A}(\mathbf{r},t),\, \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t},\, \frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial\mathbf{r}}\right] = \frac{1}{8\pi}\left\{\frac{1}{c^2}\left|\frac{\partial\mathbf{A}(\mathbf{r},t)}{\partial t}\right|^2 - |\nabla\times\mathbf{A}(\mathbf{r},t)|^2\right\} + \frac{\mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)}{c}, \tag{11}$$

where $\mathbf{A}$ is the vector potential and the (induced) current density $\mathbf{j}$ for a dispersive, nonuniform, and linear medium can be expressed as [2]

$$\mathbf{j}(\mathbf{r},t) \equiv -\frac{1}{c}\int d^3r' \int_{-\infty}^{t} dt'\,\sigma\left(\mathbf{r}-\mathbf{r}',t-t',\frac{\mathbf{r}+\mathbf{r}'}{2},\frac{t+t'}{2}\right)\cdot\frac{\partial\mathbf{A}(\mathbf{r}',t')}{\partial t'}. \tag{12}$$

Because of the integral form (12), due to the nonlocal response of the medium, the Lagrangian density is no longer simply a function of the field $\mathbf{A}$ and its derivatives, as *in vacuo*, but becomes a rather complicated functional of it. For the case for which the properties of the medium vary slowly in space and time, the kernel σ in Eq. (12) is a slow function of the sum variables, which makes it convenient to use both local Fourier transforms (the ''eikonal approximation'') and the average Lagrangian (11) over space and time. The corresponding average Lagrangian $\mathcal{L}$ can thus be expressed in terms of the Fourier transformed vector potential $\mathbf{A}(\mathbf{k},\omega,\mathbf{r},t)$ and the conductivity tensor

$$\boldsymbol{\sigma}(\mathbf{k},\omega,\mathbf{r},t) = \int d^3r' \int_0^\infty dt'\,\boldsymbol{\sigma}(\mathbf{r}',t',\mathbf{r},t)e^{-i(\mathbf{k}\cdot\mathbf{r}'-\omega t')}$$

$(=\boldsymbol{\sigma}_h + i\boldsymbol{\sigma}_a$, with $\boldsymbol{\sigma}_h$ and $\boldsymbol{\sigma}_a$, respectively, the Hermitian and the anti-Hermitian part). Limiting ourselves to a nonabsorbing medium, for which the conductivity tensor is anti-Hermitian, $\boldsymbol{\sigma} = i\boldsymbol{\sigma}_a$, and assuming that, to *zeroth order* of the geometrical optics approximation, the relation between the induced current and the electric field is satisfied locally, i.e., $\mathbf{j}(\mathbf{k},\omega,\mathbf{r},t) = i\boldsymbol{\sigma}_a(\mathbf{k},\omega,\mathbf{r},t)\cdot\mathbf{E}(\mathbf{k},\omega,\mathbf{r},t)$, there results

$$\mathcal{L}(\mathbf{A}(\mathbf{k},\omega,\mathbf{r},t),\mathbf{k},\omega,\mathbf{r},t) = \frac{1}{VT}\frac{\omega^2}{8\pi c^2}\,\mathbf{A}^*(\mathbf{k},\omega,\mathbf{r},t)\cdot\mathbf{\Lambda}(\mathbf{k},\omega,\mathbf{r},t)\cdot\mathbf{A}(\mathbf{k},\omega,\mathbf{r},t), \tag{13a}$$

where T is a time long compared to the wave period and $\mathbf{\Lambda}\equiv(c^2k^2/\omega^2)(\hat{\mathbf{k}}\hat{\mathbf{k}}-\mathbf{I})+\boldsymbol{\varepsilon}_h$ is the dispersion tensor, with $\boldsymbol{\varepsilon}_h \equiv \mathbf{I}-(4\pi/\omega)\boldsymbol{\sigma}_a$ the (Hermitian) dielectric tensor. It appears that the average Lagrangian density (13a) is a function, rather than a functional, of the field, notwithstanding the dispersive nature of the medium. On referring specifically to a given mode, the average zeroth-order Lagrangian (13a) can be written as

$$\mathcal{L}(\mathcal{A}(\mathbf{k},\omega,\mathbf{r},t),\mathbf{k},\omega,\mathbf{r},t) = \frac{1}{VT}\frac{\omega^2}{8\pi c^2}\,D(\mathbf{k},\omega,\mathbf{r},t)|\mathcal{A}(\mathbf{k},\omega,\mathbf{r},t)|^2, \tag{13b}$$

$\mathcal{A}(\mathbf{k},\omega,\mathbf{r},t)$ being the (eikonal) amplitude of the vector potential for the considered mode (which is a function of both frequency and wave vector separately, the dispersion relation being not yet available). The frequency ω and the wave vector $\mathbf{k}$, which are constants for the case of a uniform medium, are to be evaluated in terms of the derivatives of the eikonal Ψ, i.e., $\omega(\mathbf{r},t)=-\partial\Psi(\mathbf{r},t)/\partial t$ and $\mathbf{k}(\mathbf{r},t)=\partial\Psi(\mathbf{r},t)/\partial\mathbf{r}$, for the general case of a space- and time-varying medium. The average Lagrangian (13b) can be required to satisfy an *average variational principle*, in close analogy to the corresponding one that yields Maxwell's equations from the Lagrangian (11). On the basis of such an average variational principle, one gets the following Euler's equations

$$\frac{\partial\mathcal{L}}{\partial\mathcal{A}}=0\Rightarrow D(\mathbf{k},\omega,\mathbf{r},t)=0, \tag{14a}$$

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\omega}-\frac{\partial}{\partial\mathbf{r}}\cdot\frac{\partial\mathcal{L}}{\partial\mathbf{k}}=0, \tag{14b}$$

along with the ray equations of geometrical optics [2]. Equation (14a) just gives the local dispersion relation $\omega=\Omega(\mathbf{k},\mathbf{r},t)$ for the considered mode, whereas Eq. (14b), which is trivially satisfied for a uniform medium, for which $\mathcal{L}$ is independent of both position and time, can be rewritten, on exploiting the specific form (13b) of the Lagrangian density, as

$$\frac{\partial}{\partial t}\frac{\partial\mathcal{L}}{\partial\omega}+\frac{\partial}{\partial\mathbf{r}}\cdot\left(\mathbf{v}_g\frac{\partial\mathcal{L}}{\partial\omega}\right)=0, \tag{15}$$

where the equality $\partial\mathcal{L}/\partial\mathbf{k}=-\mathbf{v}_g(\partial\mathcal{L}/\partial\omega)$, with $\mathbf{v}_g\equiv-(\partial D/\partial\mathbf{k})/(\partial D/\partial\omega)$ the group velocity, has been used. The quantity $\partial\mathcal{L}/\partial\omega$ can now be explicitly calculated from Eq. (13b) and, on expressing the vector potential amplitude in terms of the electric-field amplitude, namely, $|\mathcal{A}|^2=c^2|\mathcal{E}|^2/\omega^2$, there results

$$\frac{\partial\mathcal{L}}{\partial\omega}=\frac{1}{VT}\frac{1}{\omega R_E}\frac{|\mathcal{E}(\mathbf{k},\omega,\mathbf{r},t)|^2}{8\pi}, \tag{16}$$

with R_E again the ratio of the electric to the total wave-energy density. Integrating Eq. (16) over ω, accounting for the dispersion relation, and noting that $|\mathcal{E}(\mathbf{k},\omega,\mathbf{r},t)|^2=|\mathcal{E}(\mathbf{k},\mathbf{r},t)|^2T(2\pi)\delta(\omega-\Omega(\mathbf{k},\mathbf{r},t))$ just yields the standard wave action W/Ω. It is to be noted that Eq. (15) is the same as the equation for the wave-energy density, namely,

$$\frac{\partial W}{\partial t}+\frac{\partial}{\partial\mathbf{r}}\cdot(\mathbf{v}_gW)=0, \tag{17a}$$

with the result that the corresponding ray equation for the wave-action density (16) is *not* in the form of Eq. (4), but in the form of Eq. (1). On the other hand, the equation for the spectral wave-energy density $\mathcal{W}$, such that $W=\int(d\omega/2\pi)\mathcal{W}$, obtained within the framework of the average variational approach, is quite different from Eq. (17a), namely,

$$\frac{\partial}{\partial t}(\mathcal{W}-\mathcal{L})+\frac{\partial}{\partial\mathbf{r}}\cdot(\mathbf{v}_g\mathcal{W})=-\mathcal{L}_t, \tag{17b}$$

which is encumbered by the presence of the Lagrangian density $\mathcal{L}$ and its derivative with respect to the explicit dependence on time $\mathcal{L}_t$.

The paradoxical result that a single quantity should satisfy two different equations is due to the fact that the average variational approach is a zeroth order approach, whereas, as is well known [1], to correctly obtain the transport equation for the electric-field amplitude in a space- and time-varying medium, one should consistently retain terms up to *first order* in the small parameter characterizing the nonuniformity of the medium. As for the induced current density, one has, accounting also for dissipation [1,2],

$$\mathbf{j}(\mathbf{k},\omega,\mathbf{r},t)=i\boldsymbol{\sigma}_a(\mathbf{k},\omega,\mathbf{r},t)\cdot\mathbf{E}(\mathbf{k},\omega,\mathbf{r},t)+\mathbf{K}\{\mathbf{E}\}, \tag{18a}$$

$$\mathbf{K}\{\mathbf{E}\}\equiv\boldsymbol{\sigma}_h\cdot\mathbf{E}+\frac{1}{2}[\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}_{\mathbf{k}}\boldsymbol{\sigma}_a)]\cdot\mathbf{E}+[(\boldsymbol{\nabla}\mathbf{E})^T\cdot\boldsymbol{\nabla}_{\mathbf{k}}]\cdot(\boldsymbol{\sigma}_a)^T - \frac{1}{2}\left[\frac{\partial}{\partial t}\left(\frac{\partial\boldsymbol{\sigma}_a}{\partial\omega}\right)\right]\cdot\mathbf{E}-\frac{\partial\boldsymbol{\sigma}_a}{\partial\omega}\cdot\frac{\partial\mathbf{E}}{\partial t}, \tag{18b}$$

the latter being a first-order quantity. It is just on account of this contribution that the correct energy conservation equation (17a) can be obtained [9]. On the other hand, the average Lagrangian density, obtained on the basis of Eqs. (18), would involve also terms with derivatives of the field, which makes the whole variational approach significantly more complicated to deal with.

IV. CONCLUSIONS

With reference to wave propagation in a space- and time-varying medium, the concept of wave-action density is of particular relevance. As for the standard wave-action density (3), the corresponding evolution along a geometrical optics ray is governed by the transport equation (4) (here, for simplicity, both wave emission and absorption are disregarded). Such a transport equation just states the conservation of the wave action along a ray under the condition that the continuity-type equation (5) is satisfied, this being the condition for which the radiation can be viewed as consisting of wave quanta and the description of radiation transfer in terms of the specific intensity of radiation [cf. Eqs. (6) and (7b)], applies.

As an alternative to the standard wave-action density (3), one can define generalized wave-action densities [cf. definitions (8) and (10)], which are quantities conserved along the geometrical optics ray, independently of the adiabaticity condition (5), and are such that they reduce to the standard wave action (3) in the limit of a uniform medium. As for the generalized wave-action density (10), it reduces to the standard one for media for which the adiabaticity condition (5) is satisfied. One should note, however, that with specific reference to the radiative transfer in space- and time-varying media, the usual description in terms of wave quanta rests on the concept of the standard wave-action density (3) and the utility of the generalized wave-action densities (8) and (10) remains on the whole to be assessed.

APPENDIX: RESULT (9b) FOR A UNIFORM MEDIUM

Let us write [1]

$$\int \frac{d\omega}{2\pi}\,\mathcal{W}_E(\mathbf{k},\omega,\mathbf{r},t) = \frac{1}{4\pi}\,e^*(\mathbf{k},\mathbf{r},t)\cdot\left\{\int d^3s\; e^{-i\mathbf{k}\cdot\mathbf{s}} \times\langle \mathbf{E}(\mathbf{r}+\tfrac{1}{2}\mathbf{s},t)\mathbf{E}(\mathbf{r}-\tfrac{1}{2}\mathbf{s},t)\rangle\right\}\cdot\mathbf{e}(\mathbf{k},\mathbf{r},t), \tag{A1}$$

where $\mathbf{e}(\mathbf{k},\mathbf{r},t)$ is the polarization vector of the considered mode and $\mathbf{E}(\mathbf{r},t)$ is the total electric field as a function of position and time. In the limit of uniform medium, reexpressing Eq. (A1) in terms of the Fourier-transformed electric field $\mathbf{E}(\mathbf{k},\omega)$, and noting that

$$\mathbf{E}(\mathbf{k},\omega)=\sum_\sigma \mathcal{E}^\sigma(\mathbf{k})\mathbf{e}^\sigma(\mathbf{k})(2\pi)\,\delta(\omega-\Omega^\sigma(\mathbf{k}))$$

(σ labels the mode) yield, for the considered mode,

$$\int \frac{d\omega}{2\pi}\,\mathcal{W}_E(\mathbf{k},\omega)=\frac{1}{4\pi}\int \frac{d^3k'}{(2\pi)^3}\,e^{i\mathbf{k}'\cdot\mathbf{r}} \times\mathcal{E}(\mathbf{k}+\tfrac{1}{2}\mathbf{k}')\mathcal{E}^*(\mathbf{k}-\tfrac{1}{2}\mathbf{k}'). \tag{A2}$$

If the radiation field is homogeneous, i.e., the autocorrelation tensor of the electric field is independent of position, the quantity (A2) must be independent of position, which requires $\mathbf{k}'=\mathbf{0}$ on the right-hand side of Eq. (A2), so that

$$\int \frac{d\omega}{2\pi}\,\mathcal{W}_E(\mathbf{k},\omega)=\frac{|\mathcal{E}(\mathbf{k})|^2}{4\pi V}, \tag{A3}$$

which is the result (9b). It should be emphasized that for equality (A3) to be valid, not only the medium, but also the radiation field must be uniform (the latter suffices to be spatially homogeneous).

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